

The Maximum Number of Subset Divisors of a Given Size

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1 Abstract

If s is a positive integer and A is a set of positive integers, we say that B is an s -divisor of A if $\sum_{b \in B} b \mid s \sum_{a \in A} a$. We study the maximal number of k -subsets of an n -element set that can be s -divisors. We provide a counterexample to a conjecture of Huynh that for $s = 1$, the answer is $\binom{n-1}{k}$ with only finitely many exceptions, but prove that adding a necessary condition makes this true. Moreover, we show that under a similar condition, the answer is $\binom{n-1}{k}$ with only finitely many exceptions for each s .

2 Introduction

If X is a set of positive integers, let $\sum X$ denote $\sum_{x \in X} x$. Let A be a finite subset of the positive integers. The elements of A are $a_1 < a_2 < \dots < a_n$ and let B be a subset of A . We say that B is a *divisor* of A if $\sum B \mid \sum A$. We define $d_k(A)$ to be the number of k -subset divisors of A and let $d(k, n)$ be the maximum value of $d_k(A)$ over all sets A of n positive integers.

Similarly, for $s \geq 1$ a positive integer, we say that B is an s -*divisor* of A if $\sum B \mid s \sum A$. We define $d_k^s(A)$ to be the number of k -subset s -divisors of A and let $d^s(k, n)$ be the maximum value of $d_k^s(A)$ over all sets A of n positive integers.

Note that the concepts of divisor and 1-divisor coincide. Also, if B is a divisor of A , then B is an s -divisor of A for all s , so $d_k^s(A) \geq d_k(A)$ and $d^s(k, n) \geq d(k, n)$.

Huynh [3] notes that for any values of a_1, \dots, a_{n-1} , we can pick such an a_n that any k -subset of $\{a_1, \dots, a_{n-1}\}$ will be an A -divisor. Therefore $d(k, n) \geq \binom{n-1}{k}$ for all $1 \leq k \leq n$. This motivates the definition that A is a k -*anti-pencil* if the set of k -subset divisors of A is $\binom{A \setminus \{a_n\}}{k}$. We similarly define A to be a (k, s) -*anti-pencil* if the set of k -subset s -divisors of A is $\binom{A \setminus \{a_n\}}{k}$.

Huynh [3] also formulates the following conjecture (Conjecture 22).

Conjecture 1. For all but finitely many values of k and n , $d(k, n) = \binom{n-1}{k}$.

In this paper, we provide infinite families of counterexamples, but prove that, with the exception of these families, the conjecture is true. This gives us the following modified form.

Conjecture 2. For all but finitely many integer pairs (k, n) with $1 < k < n$, $d(k, n) = \binom{n-1}{k}$.

For convenience, we now rescale, dividing every element of A by $\sum A$, so that now the elements of A are positive rational numbers and $\sum A = 1$. Under this rescaling, $B \subseteq A$ is a divisor of A if and only if $\sum B = \frac{1}{m}$ for some positive integer m and B is an s -divisor of A if and only if $\sum B = \frac{s}{m}$ for some positive integer m . Clearly, the values of $d(k, n)$ and $d^s(k, n)$ do not change.

The $k < n$ condition in Conjecture 2 is necessary since it is easy to see that $d(n, n) = 1 > \binom{n-1}{n}$. Also, if

$$A = \left\{ \frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^{n-2}}, \frac{1}{3(2^{n-1})}, \frac{1}{3(2^{n-2})} \right\}$$

then $\sum A = 1$, so $d_1(A) = n$ and $d(1, n) \geq n > \binom{n-1}{1}$. Therefore the $1 < k$ condition is necessary.

However, we prove that these families cover all but finitely many exceptions.

Theorem 3. *For all but finitely many pairs (k, n) , if $1 < k < n$, $|A| = n$, and $d_k(n) \geq \binom{n-1}{k}$, then A is a k -anti-pencil.*

Note that this immediately implies Conjecture 2.

If we are interested in s -divisors, we get another family of exceptions. If $s \geq 2$, $a_n = \frac{1}{s+1}$ and $a_{n-1} = \frac{2}{s+2}$, then $d_{n-1}^s(A) \geq 2$, so $d^s(n-1, n) \geq 2 > \binom{n-1}{n-1}$. However, we prove that these cover all but finitely many exceptions.

Theorem 4. *Fix $s \geq 1$. For all but finitely many pairs (k, n) (with the number of these pairs depending on s), if $1 < k < n-1$, $|A| = n$, and $d_k^s(n) \geq \binom{n-1}{k}$, then A is a (k, s) -anti-pencil.*

Note that this immediately implies the following corollary.

Corollary 5. *Fix $s \geq 1$. Then $d^s(k, n) = \binom{n-1}{k}$ for all but finitely many pairs (k, n) with $1 < k < n-1$ (with the number of these pairs depending on s).*

We will prove Theorem 4. In the $s = 1$ case, where $k = n-1$, if $i \leq n-1$, then $\sum(A \setminus \{a_i\}) > \frac{1}{2}$, so $A \setminus \{a_i\}$ is not a divisor of A . This, together with the $s = 1$ case of Theorem 4, gives us Theorem 3.

3 Lemmas

Take a d -dimensional lattice cube with n lattice points per edge. Define a poset on the lattice points by $(x_1, \dots, x_d) \leq (y_1, \dots, y_d)$ if $x_i \leq y_i$ for all i .

Lemma 6. *The largest antichain in this poset has at most $(n + d - 2)^{d-1} \sqrt{\frac{2}{d}}$ elements.*

Proof. First, we need some definitions.

The *width* of a poset is the size of its largest antichain. If P is a finite poset, we say that P is *ranked* if there exists a function $\rho : P \rightarrow \mathbb{Z}$ satisfying $\rho(y) = \rho(x) + 1$ if y covers x in P (i.e. $y > x$, and there is no $z \in P$ with $y > z > x$). If $\rho(x) = i$, then x is said to have *rank* i . Let P_i denote the set of elements of P of rank i . We say P is *rank-symmetric rank-unimodal* if there exists some $c \in \mathbb{Z}$ with $|P_i| \leq |P_{i+1}|$ when $i < c$ and $|P_{2c-i}| = |P_i|$ for all $i \in \mathbb{Z}$. A ranked poset P is called *strongly Sperner* if for any positive integer s , the largest subset of P that has no $(s + 1)$ -chain is the union of the s largest P_i .

Proctor, Saks, and Sturtevant [6] prove that the class of rank-symmetric rank-unimodal strongly Sperner posets is closed under products.

Since a linear ordering of length n is rank-symmetric rank-unimodal strongly Sperner, so is a product of d of them (the lattice cube).

Center the cube on the origin by translation in \mathbb{R}^d . Let U be the set of elements whose coordinates sum to 0. Since the poset is rank-symmetric rank-unimodal strongly Sperner, its width is at most the size of P_c , which is $|U|$.

For each $y = (y_1, \dots, y_d) \in U$, let S_y be the set of points (x_1, \dots, x_d) with $|x_i - y_i| < \frac{1}{2}$ for $1 \leq i \leq d-1$ (note that this does not include the last index) which lie on the hyperplane given by $x_1 + \dots + x_d = 0$. If y, z are distinct elements of U , then S_y and S_z are clearly disjoint. Also, the projection of S_y onto the hyperplane given by $x_d = 0$ is a unit $(d-1)$ -dimensional hypercube, which has volume 1. Thus the volume of S_y is \sqrt{d} and the volume of $\bigcup_{y \in U} S_y$ is $|U|\sqrt{d}$.

On the other hand, if $(x_1, \dots, x_d) \in S_y$, then $|x_i - y_i| < \frac{1}{2}$ for $1 \leq i \leq d-1$ and $|x_d - y_d| \leq \sum_{i=1}^{d-1} |x_i - y_i| < \frac{1}{2}(d-1)$. Thus (x_1, \dots, x_d) lies in the cube of edge length $(n-1) + (d-1) = n + d - 2$ centered at the origin. Therefore $\bigcup_{y \in U} S_y$ lies in the intersection of a cube of edge length $n + d - 2$ with a hyperplane through its center (the origin).

Ball [1] shows that the volume of the intersection of a unit hypercube of arbitrary dimension with a hyperplane through its center is at most $\sqrt{2}$. Therefore the volume of $\bigcup_{y \in U} S_y$ is at most $(n + d - 2)^{d-1} \sqrt{2}$, so

$$|U| \leq (n + d - 2)^{d-1} \sqrt{\frac{2}{d}}.$$

□

Let $X = \{x_1 < \dots < x_n\}$ be any set of positive integers. If $B, C \in \binom{X}{d}$, then we say that $B \leq C$ if we can write $B = \{b_1, \dots, b_d\}$ and $C = \{c_1, \dots, c_d\}$ with $b_i \leq c_i$ for all $1 \leq i \leq d$. Whenever we compare subsets of A , we will be using this partial order.

Lemma 7. *Fix $d > 1$. For n sufficiently large, the width of the partial order defined above is less than $\frac{2}{\sqrt{d}} \frac{1}{n} \left| \binom{X}{d} \right|$.*

Proof. Let U be a maximum antichain of the partial order. Take the partial order of X^d , which coincides with the cube partial order. Let $U' = \{(y_1, \dots, y_d) \in X^d \mid \{y_1, \dots, y_d\} \in U\}$. Note that this means, in particular, that all elements of any k -tuple in U' are distinct. If $(y_1, \dots, y_d), (z_1, \dots, z_d) \in U'$ with $(y_1, \dots, y_d) < (z_1, \dots, z_d)$, then we get that $\{y_i\} \leq \{z_i\}$ and $\sum_{i=1}^d y_i < \sum_{i=1}^d z_i$, so $\{y_i\} \neq \{z_i\}$, so $\{y_i\} < \{z_i\}$, which is impossible. Thus U' is an antichain of X^d of size $d!|U|$ and

$$|U| \leq \frac{1}{d!}(n+d-2)^{d-1} \sqrt{\frac{2}{d}}.$$

Then $\left| \binom{X}{d} \right| = \binom{n}{d}$ gives us

$$\frac{|U|}{\left| \binom{X}{d} \right|} \leq \frac{(n+d-2)^{d-1} \sqrt{\frac{2}{d}}}{n(n-1) \cdots (n-d+1)}.$$

For sufficiently large n , $\left(\frac{n+d-2}{n-d+1} \right)^{d-1} < \sqrt{2}$, so $\frac{|U|}{\left| \binom{X}{d} \right|} < \frac{2}{\sqrt{d}} \frac{1}{n}$. □

Let $d(n)$ denote the number of divisors of n .

Lemma 8. *For any positive integer k , $d(n) = O(n^{\frac{1}{k}})$.*

Proof. There are finitely many primes $p < 2^k$, so there must be some constant C such that for any $p < 2^k$ and any positive integer m , $d(p^m) = m+1 \leq C(p^m)^{\frac{1}{k}}$.

For $p > 2^k$, $d(p^m) = m+1 \leq 2^m \leq (p^m)^{\frac{1}{k}}$. Thus if $n = \prod_{i=1}^j p_i^{m_i}$ for distinct prime p_i , then

$$d(n) = \prod_{i=1}^j d(p_i^{m_i}) \leq C^{2^k} \prod_{i=1}^j (p_i^{m_i})^{\frac{1}{k}} \leq C^{2^k} n^{\frac{1}{k}} = O\left(n^{\frac{1}{k}}\right).$$

□

Lemma 9. *Fix positive integers k, m, a, b . Then for positive integers n , the number of pairs of positive integers (x, y) such that $\frac{m}{n} = \frac{a}{x} + \frac{b}{y}$ and all three fractions are in lowest terms is at most $O(n^{\frac{1}{k}})$.*

Proof. Assume $\frac{m}{n} = \frac{a}{x} + \frac{b}{y}$. Let $p = \gcd(n, x)$, with $n = tp$ and $x = wp$. Then

$$\frac{b}{y} = \frac{m}{n} - \frac{a}{x} = \frac{mw - at}{twp}.$$

Letting $q = \gcd(mw - at, twp)$, we get

$$mw - at = qb. \tag{1}$$

For any choice of n, p, q , (1) gives at most one possible value of w , thus at most one value of x , and thus at most one value of (x, y) .

The definition of q gives us $q \mid p$. Then for a given n , both p and q are divisors of n , so by Lemma 8 there are $O(n^{\frac{1}{2k}})$ possible values for p and $O(n^{\frac{1}{2k}})$ values for q , so there are $O(n^{\frac{1}{k}})$ values for (p, q) and $O(n^{\frac{1}{k}})$ pairs of numbers (x, y) . □

4 Proof of Theorem 4

Assume that $|A| = n$, $d_k^s(A) \geq \binom{n-1}{k}$, and that A is not a (k, s) -anti-pencil. Note that then some $B \ni a_n$ has $\sum B \leq \frac{s}{s+1}$, so since $1 < k$, we have $a_n < \frac{s}{s+1}$. We will use this in all the cases below. Also, the number of k -subsets of A that are not s -divisors is at most $\binom{n}{k} - \binom{n}{k-1} = \binom{n-1}{k-1}$.

Remark 10. If B and C are k -subsets of A with $B < C$, then $\sum B < \sum C$. Note that if $B_0 < B_1 < \dots < B_m$ are all divisors of A and $\sum B_m < s/q$, then $\sum B_0 < s/(q+m)$. Therefore if $a \in B_0$, then $a < s/(q+m)$. Since $k < n$, $\sum B_m < s/s$, so we automatically get that $a < s/(s+m)$.

Each of the subsections below is a separate case.

4.1 k small

Fix $2 \leq k$ and let $n \gg k$.

For $1 \leq i_1, \dots, i_k \leq n$, call the ordered k -tuple (i_1, \dots, i_k) *repetitive* if not all entries are distinct. Call it *good* if all entries are distinct and $\{a_{i_j}\}$ is an s -divisor. Otherwise, call the ordered k -tuple *bad*.

We will first restrict our attention to k -tuples where $i_k \geq n-1$. Among these, $O(n^{k-2})$ are repetitive. Also, $O(n^{k-2})$ include both n and $n-1$ among their components. Of the remainder, at most $(k-1)! \binom{n-1}{k-1} \leq n^{k-1}$ are bad. Thus at least $1/3$ of the k -tuples (i_1, \dots, i_k) satisfying $i_k \geq n-1$ are good.

By the Pigeonhole Principle, there are some values j_2, \dots, j_k with $j_k \geq n-1$ such that the chain $\{(1, j_2, \dots, j_k), \dots, (n, j_2, \dots, j_k)\} \subset U$ has at least $n/3$ good k -tuples. This gives us a chain of k -subset s -divisors of length at least $n/3$. Thus $a_{n-1} \leq \frac{3s}{n}$.

Let $B = \{a_i \mid i > (1 - \frac{1}{9s^2})n\}$. If $a_i \in B$, then

$$1 = \sum A = \sum_{i=1}^n a_i < na_i + \frac{n}{9s^2} a_{n-1} + a_n < na_i + \frac{1}{3s} + \frac{s}{s+1}$$

so $na_i > \frac{1}{6s}$ and $a_i > \frac{1}{6sn}$.

Thus any s -divisors that is a subset of B must sum to some $\frac{s}{m} > \frac{1}{6sn}$, so there are at most $6s^2n$ distinct values that m can take. Thus there are at most $6s^2n$ distinct values that an s -divisor that is a subset of B can sum to.

If $D \in \binom{B}{k-2}$ and $r = \frac{s}{m}$ for some positive integer m , call D an r -stem if there are at least $\frac{1}{10000s^6}n$ pairs $\{x, y\} \subset B \setminus D$ with $\sum(D \cup \{x, y\}) = r$. Call such pairs *tails* of D . If two tails of D are $\{x, y\}$ and $\{x, z\}$, then the sum condition gives us $y = z$, so tails of D are pairwise disjoint.

Now let $B_0 = B$. Note that $|B_0| > \frac{1}{10s^2}n$. As long as $|B_{i-1}| \geq \frac{1}{20s^2}n$, B_{i-1} has at most $\binom{n-1}{k-1}$ subsets which are not s -divisors of A , so it has at least $\frac{1}{2} \binom{|B_{i-1}|}{k}$ k -subsets that are s -divisors. Since these take on at most $6s^2n$ values, there must be some positive integer m_i such that at least $\frac{1}{12s^2n} \binom{|B_{i-1}|}{k}$ k -subsets of B_{i-1} sum to $r_i = \frac{s}{m_i}$.

If we randomly choose $D_i \in \binom{B_{i-1}}{k-2}$, the expected value for the number of pairs $\{x, y\} \subset B_{i-1} \setminus D_i$ with $\sum(D_i \cup \{x, y\}) = r_i$ is at least $\frac{1}{12s^2n} \binom{|B_{i-1}| - (k-2)}{2}$. Thus we will choose a D_i such that the number of these pairs is at least $\frac{1}{12s^2n} \binom{|B_{i-1}| - (k-2)}{2}$. Since

$$\frac{1}{12s^2n} \binom{|B_{i-1}| - (k-2)}{2} \geq \frac{1}{25s^2n} (|B_{i-1}|)^2 \geq \frac{1}{25s^2n(20s^2)^2} n^2 \geq \frac{1}{10000s^6} n,$$

D_i satisfies the definition of an r_i -stem.

Let $B_i = B_{i-1} \setminus D_i$. Then for $i \leq \frac{1}{20ks^2}n$, D_i is an r_i -stem and all the D_i are disjoint.

Since the number of k -subsets of A which are not s -divisors is less than $\binom{\frac{1}{20ks^2}n}{k}$, we know that there must exist disjoint D_{i_1}, \dots, D_{i_k} such that any set consisting of one element of each D_{i_j} will be an s -divisor. Note that in the $k = 2$ case, $D_{i_1} = D_{i_2} = \emptyset$. Partition $\bigcup_{j=1}^k D_{i_j}$ into $k - 2$ such sets C_1, \dots, C_{k-2} .

Let $p = \lceil \frac{1}{10000s^6}n / (2k) \rceil = \lceil \frac{1}{20000s^6k}n \rceil$. For $1 \leq j \leq k$, we want to choose T_1^j, \dots, T_p^j to be tails of D_{i_j} . We will choose them for $j = 1$, then for $j = 2$, and so on. When we choose $\{T_\ell^j\}$, we will make each of these tails disjoint from each of the k stems, as well as from the already chosen tails. This is possible since

$$\left| \bigcup_{h=1}^k D_{i_h} \cup \bigcup_{h=1}^{j-1} \bigcup_{\ell=1}^p T_\ell^h \right| = \left| \bigcup_{h=1}^k D_{i_h} \right| + \sum_{h=1}^{j-1} \left| \bigcup_{\ell=1}^p T_\ell^h \right| = k(k-2) + 2(j-1)p \leq k(k-2) + 2(k-1)p.$$

Since any element in a stem or in a previously chosen tail can be in at most one tail of D_{i_j} , at most $k(k-2) + 2(k-1)p$ tails are eliminated, so there must be at least p tails still available to choose from.

We say that a choice of k tails $\{T_{i_j}^j\}_{j=1}^k$ for each stem is *fortuitous* if $\{x_{i_j}^j\}_{j=1}^k$ and $\{y_{i_j}^j\}_{j=1}^k$ are both s -divisors. There are $p^k > n^k / (20000s^6k)^k$ choices of tails, and at most $\binom{n-1}{k-1}$ of them are not fortuitous. Thus at least $\frac{1}{2}$ of possible choices are fortuitous.

By the Pigeonhole Principle, we can choose i_1, \dots, i_{k-1} so that there are at least $p/2$ choices for i which make $\{T_{i_1}^1, \dots, T_{i_{k-1}}^{k-1}, T_i^k\}$ fortuitous.

Note that different choices of i give us different values of x_i^k and therefore different values of $\sum_{j=1}^k x_{i_j}^j$, so $\sum_{j=1}^k x_{i_j}^j$ can take on at least

$$p/2 = \Omega(n)$$

different values.

On the other hand, if we are given a fortuitous choice of tails $\{T_{i_j}^j\}$, then

$$\begin{aligned} \sum_{j=1}^{k-2} \sum C_j + \sum_{j=1}^k x_{i_j}^j + \sum_{j=1}^k y_{i_j}^j &= \sum_{j=1}^k \sum \left(D_{i_j} \cup \{x_{i_j}^j, y_{i_j}^j\} \right) \\ \sum_{j=1}^k x_{i_j}^j + \sum_{j=1}^k y_{i_j}^j &= \sum_{j=1}^k r_{i_j} - \sum_{j=1}^{k-2} \sum C_j. \end{aligned}$$

The right hand side does not depend on our choice of tails. Also, since each r_{i_j} and each $\sum C_j$ has denominator at most $6s^2n$, the right hand side has denominator at most $(6s^2n)^{2k}$. Since both $\sum_{j=1}^k x_{i_j}^j$ and $\sum_{j=1}^k y_{i_j}^j$ are s -divisors, there are at most s^2 possibilities for their numerators. For each such possibility, by Lemma 9, $\sum_{j=1}^k x_{i_j}^j$ can take on at most

$$O\left((6s^2n)^{2k}\right)^{\frac{1}{4k}} = O\left(sn^{\frac{1}{2}}\right)$$

different values. Thus $\sum_{j=1}^k x_{i_j}^j$ can take on at most

$$O\left(s^3n^{\frac{1}{2}}\right)$$

different values, contradicting the upper bound above.

4.2 $n \geq \frac{3}{2}k$, k sufficiently large

Let $d = \lceil (s(s+1)/0.03)^2 \rceil$. Assume that k is sufficiently large relative d and that $n \geq \frac{3}{2}k$.

Let T_2 be the set of k -subsets of A that include both a_{n-1} and a_n . Let T_1 be the set of k -subsets of A that include one of a_{n-1} or a_n , but not both. Define U_1 and U_2 similarly, but with $(k-d)$ -subsets.

For $S \in U_t$, let $P_S = \{B \in T_t \mid S \subset B\}$ (the set of k -subsets obtainable by adding d elements of A less than a_{n-1} to S). Note that an element of T_t is contained in P_S for exactly $\binom{k-t}{d}$ values of S . Thus if $\alpha|T_t|$ elements of T_t are s -divisors, then there is some $S \in U_t$ so that at least $\alpha|P_S|$ elements of P_S are s -divisors.

Now note that the disjoint union $T_1 \cup T_2$ is the set of all k -subsets whose greatest element is at least a_{n-1} , so

$$|T_1 \cup T_2| = \binom{n}{k} - \binom{n-2}{k}$$

and the fraction of the elements of $T_1 \cup T_2$ which are not s -divisors is at most

$$\begin{aligned}
\frac{\binom{n}{k} - \binom{n-1}{k}}{\binom{n}{k} - \binom{n-2}{k}} &= \frac{1}{\frac{\binom{n}{k} - \binom{n-2}{k}}{\binom{n-1}{k-1}}} \\
&= \frac{1}{\frac{n}{k} - \frac{\binom{n-2}{k}}{\binom{n-1}{k-1}}} \\
&= \frac{1}{\frac{n(n-1)}{k(n-1)} - \frac{(n-k)(n-k-1)}{k(n-1)}} \\
&= \frac{k(n-1)}{n(n-1) - (n-k)(n-k-1)} \\
&= \frac{k(n-1)}{k(2n-1-k)} \\
&= \frac{n-1}{2n-k-1} \\
&\leq 0.76
\end{aligned}$$

for sufficiently large k . Therefore, if we set $\alpha = 0.24$, then for $t = 1$ or $t = 2$, the fraction of elements of T_t that are s -divisors is at least α , so for some S , the fraction of elements of P_S which are s -divisors is at least $\alpha = 0.24$.

Note that the partial order of P_S is the same as the partial order of $\binom{A \setminus S \setminus \{a_{n-1}, a_n\}}{d}$, so by Lemma 7, its width is at most $\frac{2}{\sqrt{d}} \frac{1}{n-k-2} |P_S|$. Then, by Mirsky's theorem, there is a chain of k -subset s -divisors in P_S of length at least

$$\frac{\alpha |P_S|}{\frac{2}{\sqrt{d}} \frac{1}{n-k-2} |P_S|} = 0.12\sqrt{d}(n-k-2) \geq (0.03\sqrt{d})n.$$

But then the first element of the chain includes a_{n-1} or a_n , so by Remark 10, $a_{n-1} \leq \frac{s}{(0.03\sqrt{d})n}$. Then

$$\sum_{i=1}^{n-1} a_i \leq \frac{s}{0.03\sqrt{d}}$$

and, since $a_n < \frac{s}{s+1}$,

$$\sum_{i=1}^n a_i < 1$$

yielding a contradiction.

4.3 $\frac{2}{3}n < k < n - (6s^2 + 3s)^2$, k sufficiently large

Let $d = (6s^2 + 3s)^2$. Assume that k is sufficiently large and that $\frac{2}{3}n < k < n - d$.

Randomly arrange the elements of A around a circle. Let M be the set of k -subsets of A consisting of k consecutive elements around the circle, and let $N = \{B \in M \mid \sum B \leq \frac{1}{2(s+1)}\}$. If $B, C \in N$ and they are shifted relative each other by at least $n-k-1$, then $|A \setminus (B \cup C)| \leq 1$, so

$$\sum A \leq \sum (A \setminus (B \cup C)) + \sum B + \sum C < \frac{s}{s+1} + \frac{1}{2(s+1)} + \frac{1}{2(s+1)} = 1,$$

which is impossible.

Thus any two elements of N are shifted by at most $n-k-2$ around the circle. This gives us $|N| \leq n-k-1$. Since any k -subset of A summing to at most $\frac{1}{2(s+1)}$ has equal probability of being in N , this tells us that the number of k -subsets with sum at most $\frac{1}{2(s+1)}$ is at most $\frac{n-k-1}{n} \binom{n}{k}$. Thus there are at least

$$\binom{n-1}{k} - \frac{n-k-1}{n} \binom{n}{k} = \frac{n-k}{n} \binom{n}{k} - \frac{n-k-1}{n} \binom{n}{k} = \frac{1}{n} \binom{n}{k}$$

k -subsets which are s -divisors of A and have a sum of elements greater than $\frac{1}{2(s+1)}$. The sum of elements of such a set is $\frac{s}{m} > \frac{1}{2(s+1)}$, so it can take on one of $2s(s+1) - s = 2s^2 + s$ values, so there must be some integer m so that at least $\frac{1}{(2s^2+s)n} \binom{n}{k}$ of the k -subsets of A sum to $\frac{s}{m}$. Thus at least $\frac{1}{(2s^2+s)n} \binom{n}{n-k}$ of the $(n-k)$ -subsets of A sum to $1 - \frac{s}{m}$.

If $S \in \binom{A}{n-k-d}$, let P_S be the set of $(n-k)$ -subsets obtainable by adding d elements of A to S . Note that any $(n-k)$ -subset of A is contained in P_S for exactly $\binom{n-k}{d}$ values of S , so there is some S so that at least

$$\frac{1}{(2s^2+s)n} |P_S|$$

elements of P_S sum to $1 - \frac{s}{m}$. They must then form an antichain.

However, the partial order of P_S is the same as the partial order of $\binom{A \setminus S}{d}$, so by Lemma 7, its largest antichain has size less than

$$\frac{2}{\sqrt{d}} \frac{1}{k+d} |P_S| < \frac{3}{n\sqrt{d}} |P_S| \leq \frac{1}{(2s^2+s)n} |P_S|,$$

yielding a contradiction.

4.4 $n - (6s^2 + 3s)^2 \leq k < n - 1$, k sufficiently large

Assume that $n - (6s^2 + 3s)^2 \leq k < n - 1$. Let $u = n - k$. Thus $1 < u \leq (6s^2 + 3s)^2$, so u can take on only finitely many values. Assume that k is sufficiently large relative those values. Let

$$Y = \left\{ B \in \binom{A}{u} \mid A \setminus B \text{ is an } s\text{-divisor of } A \right\}.$$

By assumption, $|Y| \geq \binom{n-1}{k} = \binom{n-1}{u-1}$.

Let q be as small as possible so that $a_{n-q} < \frac{1}{u(s+1)}$. Note that $q < u(s+1)$. If $B \in \binom{A}{u}$ and $b \leq a_{n-q}$ for all $b \in B$, then $\sum B < \frac{1}{s+1}$, so $\sum(A \setminus B) > \frac{s}{s+1}$ and $B \notin Y$. Thus every $B \in Y$ contains at least one of the q greatest elements of A .

The number of u -subsets of A containing at least 2 of the q greatest elements of A is bounded by

$$2^q \binom{n-q}{u-2} < 2^{u(s+1)} \binom{n}{u-2} < \frac{1}{2}|Y|,$$

so at least half of the elements of Y contain exactly one of the q greatest elements of A .

Thus there must be some a_i which is one of the q greatest elements of A such that at least $\frac{1}{2u(s+1)} \binom{n-1}{u-1}$ elements of Y include a_i and no other of the q largest elements.

If B is such an element of Y , then

$$\sum B < a_i + (u-1) \frac{1}{u(s+1)} < \frac{s}{s+1} + \frac{u-1}{u(s+1)} = 1 - \frac{1}{u(s+1)}.$$

Since $\sum B$ must be of the form $1 - \frac{s}{m}$ for some positive integer m , we get fewer than $s(s+1)u$ possible values of m . Thus there must be some value of m so that there are at least

$$\frac{1}{2u^2s(s+1)^2} \binom{n-1}{u-1}$$

different u -subsets of A which include a_i and sum to $1 - \frac{s}{m}$. However, if we have a collection of that many u -subsets of A that contain a_i , then some 2 of them will share $u-1$ elements and thus have different sum. This gives us a contradiction.

5 Conclusion

For k sufficiently large, all n are covered by one of the three last cases. For k small, all but finitely values of n are covered by the first case.

In the statement of Theorem 3 and Theorem 4, “all but finitely many” cannot be omitted. For example, Huynh[3] notes that $n = 4, k = 2, A = \{\frac{1}{24}, \frac{5}{24}, \frac{7}{24}, \frac{11}{24}\}$ gives $d_k(A) = 4 > \binom{n-1}{k}$. As s increases, the number of such exceptions grows; in fact, it is easy to see that any n, k, A , will be an exception for sufficiently large s .

We could follow the proof and trace out the upper bounds on n such that (k, n) is an exception; however these will probably be far from optimal (for instance, for $s = 1$, $(2, 4)$ is likely the only exception). It would be interesting to get a good bound on the number of such exceptions, or on how large n can be in terms of s for (k, n) to be an exception.

In this paper, we are counting $B \in \binom{A}{k}$ such that $\sum B = \frac{s}{m}$. If we instead counted B such that $\sum B < \frac{k}{n}$, this problem becomes equivalent to the Manickam-Miklós-Singhi conjecture:

Conjecture 11. For positive integers n, k with $n \geq 4k$, every set of n real numbers with nonnegative sum has at least $\binom{n-1}{k-1}$ k -element subsets whose sum is also nonnegative.

The equivalence is given by taking the complement of B and applying a linear transformation.

The MMS conjecture has been proven for $k \mid n$ [4], $n \geq 10^{46}k$ [5], and $n \geq 8k^2$ [2], however there are pairs (n, k) such that it does not hold. This suggests a more general problem.

Problem 12. Fix $S \subseteq [0, 1]$ and positive integers n and k . If A is a set of positive reals, let $d_k(S, A)$ be the number of subsets $B \in \binom{A}{k}$ such that $\sum B = S$. Let $d(S, k, n)$ be the maximal value of $d_k(S, A)$ over all A with $|A| = n$ and $\sum A = 1$. For what S, k, n do we get $d(S, k, n) = \binom{n-1}{k}$? Furthermore, when does $d_k(S, A) \geq \binom{n-1}{k}$ imply that A is an k -anti-pencil?

This paper addresses this problem for $S = \{\frac{s}{m} \mid m \in \mathbb{Z}^+\}$, while the MMS conjecture deals with this problem for $S = (0, k/n)$. Another example of a set for which this problem might be interesting is a set of the form $S = (0, \alpha k/n) \cup \{\frac{s}{m} \mid m \in \mathbb{Z}^+\}$, which combines the theorem of this paper with the MMS conjecture.

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References

- [1] K. Ball. Cube slicing in \mathbb{R}^n . *Proc. Amer. Math. Soc.*, 97(3):465–472, 1986.
- [2] A. Chowdhury, G. Sarkis, and S. Shahriari. A new quadratic bound for the Manickam-Miklós-Singhi conjecture. 2014. preprint, <http://arxiv.org/abs/1403.1844>.
- [3] Tony Huynh. Extremal problems for subset divisors. *Elect. J. Combin.*, 21:P1.42, 2014.
- [4] N. Manickam and D. Miklós. On the number of nonnegative partial sums of a nonnegative sum. In *Combinatorics (Eger, 1987)*, Colloq. Math. Soc. János Bolyai 52 (North-Holland, 1988), 385–392.
- [5] A. Pokrovskiy. A linear bound on the Manickam-Miklos-Singhi conjecture. 2013. preprint, <http://arxiv.org/abs/1308.2176>.
- [6] R. A. Proctor, M. Saks, and D. Sturtevant. Product partial orders with the Sperner property. *Discrete Math.*, 30:173–180, 1980.